THE SPECTRUM OF A MAGNETIC SCHRÖDINGER OPERATOR WITH RANDOMLY LOCATED DELTA IMPURITIES

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Abstract

We consider a single band approximation to the random Schrödinger operator in an external magnetic field. The spectrum of such an operator has been characterized in the case where delta impurities are located on the sites of a lattice. In this paper we generalize these results by letting the delta impurities have random positions as well as strengths; they are located in squares of a lattice with a general bounded distribution. We characterize the entire spectrum of this operator when the magnetic field is sufficiently high. We show that the whole spectrum is pure point, the energy coinciding with the first Landau level is infinitely degenerate and that the eigenfunctions corresponding to other Landau band energies are exponentially localized.

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I. Introduction

In recent years there has been considerable activity in the study of random magnetic Schrödinger operators mainly due to their relation with the theory of the Integer Quantum Hall Effect (IQHE). Some of these studies have incorporated the randomness into the magnetic field[1], whereas others have added a random potential to the usual Landau Hamiltonian. Without any disorder the Landau Hamiltonian has a spectrum of evenly spaced *Landau levels*, each one of which is an infinitely degenerate eigenenergy. When a random potential is added these Landau levels broaden into bands. In several models[2–4] it has been shown that for large magnetic field the spectrum at the edges of the bands is pure point, with each eigenenergy corresponding to an exponentially localized state. The proofs rely on von Dreifus and Klein's[5] refined version of the earlier multiscale analysis by Fröhlich and Spencer[6] and on percolation theory. These results are not sufficient to provide a complete understanding of the IQHE however, as the nature of the spectrum in the interior of the band is crucial in explaining the observed plateaux[7]. In one special case[8,9] the spectrum has been completely characterized. In this work the authors consider a random potential consisting of zero-range scatterers (delta functions) situated on the sites of a regular lattice. In the first paper[8], they show that, in the case of a single-band approximation, the whole spectrum is pure point, with exponentially localized states for all energies except the original Landau level. They prove also that this level remains infinitely degenerate. These results are improved in a later work[9] where they obtain similar results for the unprojected Hamiltonian. They adopt a simple proof of localization by Aizenman and Molchanov[10] which utilizes low moments of the resolvent kernel.

The purpose of this work is to generalize the above for the case of a magnetic Schrödinger operator with randomly distributed delta impurities. Specifically, the random potential consists of point scatterers, delta functions, positioned in unit squares which are centered on the Gaussian integers, so that it is possible to have up to four scatterers arbitrarily close together. The strengths of the scatterers will also be random. We consider a two-dimensional infinite system of noninteracting electrons moving in a uniform magnetic field of strength B and the random potential V. The precise hypotheses on the probability distributions will be stated in Section II.

In the symmetric gauge the vector potential is given by $\mathbf{A}(\mathbf{r}) = \frac{1}{2}(\mathbf{r} \times \mathbf{B})$ and the Hamiltonian is

$$H = (-i\nabla - \mathbf{A}(\mathbf{r}))^2 + V(\mathbf{r}). \tag{1.1}$$

When the magnetic field is sufficiently strong in comparison to the random potential the Landau bands do not overlap and it is sufficient to consider the projection of the Hamiltonian onto only one of them. The Hamiltonian retricted to the nth level is

$$H_n = B(2n+1)P_n + P_n V P_n, \tag{1.2}$$

where P_n denotes the projection onto the level. The first term comes from the decomposition of the purely kinetic part of (1.1) and can be dropped as it modifies the energy only by a constant. Note that the resulting Hamiltonian is a random integral operator instead of a differential operator and that the kernels of P_n are known explicitly. For simplicity, in this paper, we restrict ourselves to the case $n = 0$ but the case $n \neq 0$ can be treated similarly.

For our model, in the special case where the support of the positional probability distribution is bounded within each unit box so that a corridor exists between impurities, the method of Aizenman and Molchanov yields a simple proof of localization[11]. However, for general distributions of position, impurities can become arbitrarily close to each other and we are not able to use their method. This is partly due to possible resonances; that when impurities can become arbitrarily close together the low moments of the resolvent kernel do not converge rapidly enough. In this paper we use the modification of the Theorem of von Dreifus and Klein given in Ref. 2 to show exponential localization of states corresponding to each of the eigenenergies separately (except the original Landau level eigenenergy). We do this by studying, at fixed energy, the behaviour of the generalized eigenfunctions at the impurity sites only, thus reducing the problem to the study of a random matrix. The eigenfunctions of this matrix are related to the eigenfunctions of the Hamiltonian in such a way that exponential decay of the former implies exponential decay of the latter. Then using Kotani's 'trick' [12] we can show exponential decay for all eigenenergies in an interval with probability one implying that the whole spectrum is pure-point. That the original Landau level eigenenergy remains infinitely degenerate has been shown in Ref. 13 for the case of a Poisson distribution of impurities. The result given here is similar and so only a sketch of the proof is given.

The paper is organised as follows. In Section II we give a precise definition of the model. In Section III we characterize the spectrum as a set, state the main theorem and show infinite degeneracy of the original eigenenergy. Also in this section we relate the Hamiltonian to a lattice operator and state our version of the adapted von Dreifus-Klein Theorem. Section IV contains the main work of this paper, where the conditions of the main theorem are

checked. In Section V we use Kotani's trick to show exponential decay and pure-point spectrum with probability one.

II. Definition and Boundedness of the Hamiltonian

Let ω_n, $n \in \mathbb{Z}[i] \equiv \{n_1 + in_2 : \ (n_1, n_2) \in \mathbb{Z}^2\}$, the Gaussian integers, be independent, identically distributed (i.i.d) random variables representing the strengths of the impurities. We shall assume that their distribution is given by an absolutely continuous probability measure μ whose support is a compact interval $X = [a, b] \subset \mathbb{R}$ containing the origin and whose density ρ is bounded by a constant ρ_b. We let $\Omega_1 = X^{\mathbb{Z}[i]}$ and $\mathbb{P}_1 = \prod_{n\in\mathbb{Z}[i]} \mu$.

Define the unit squares centred at $n \in \mathbb{Z}[i]$:

$$B_n = \{z \in \mathbb{R}^2 \,|\, n_i - \frac{1}{2} \leq z_i < n_i + \frac{1}{2}\,,\, n \in \mathbb{Z}[i]\,,\, i = 1, 2\,\}.$$

Let $\zeta_n = n + \tilde{\zeta}_n$, $n \in \mathbb{Z}[i]$, represent the positions of the impurities in the complex plane. $\tilde{\zeta}_n$, $n \in \mathbb{Z}[i]$ are i.i.d. random variables. We shall assume that their distribution is given by a probability measure ν with support equal to B_0 and density r bounded by a constant r_b. We let $\Omega_2 = \times_{n\in\mathbb{Z}[i]} B_0$ and $\mathbb{P}_2 = \prod_{n\in\mathbb{Z}[i]} \nu$. Our probability space will be $\Omega = \Omega_1 \times \Omega_2$ with probability measure $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$.

For $m \in \mathbb{Z}[i]$ let τ_m be the measure preserving automorphism of Ω corresponding to translation by m:

$$(\tau_m(\omega, \zeta))_n = (\omega_{n-m}, \zeta_{n-m}) \ . \tag{2.1}$$

The group $\{\tau_m : m \in \mathbb{Z}[i]\}$ is ergodic for the probability measure $\mathbb{P}$.

Let $\mathcal{H} = L^2(\mathbb{C})$ and let $\mathcal{H}_0$ be the eigenspace corresponding to the lowest eigenvalue (first

Landau level) of the kinetic part of the Hamiltonian defined in (1.1) and let P_0 be the orthogonal projection onto $\mathcal{H}_0$. The Hamiltonian for our model is the operator on $\mathcal{H}_0$ given formally by

$$H(\omega,\zeta) = \frac{\pi}{2\kappa} P_0 V(\cdot,(\omega,\zeta)) = \frac{\pi}{2\kappa} P_0 V(\cdot,(\omega,\zeta)) P_0 \tag{2.2}$$

where $(\omega,\zeta) \in \Omega$ and

$$V(z,(\omega,\zeta)) = \sum_{n\in\mathbb{Z}[i]} \omega_n \delta(z-\zeta_n). \tag{2.3}$$

Note that H coincides with H_0 in (1.2) up to the term BP_0 and a multiplicative constant and that the lowest Landau energy is now shifted to zero. The projection P_0 is an integral operator with kernel

$$P_0(z,z') = \frac{2\kappa}{\pi} \exp[-\kappa|z-z'|^2 - 2i\kappa z \wedge z'], \tag{2.4}$$

where $\kappa = B/4$ and $z \wedge z' = \mathcal{R}z\mathcal{I}z' - \mathcal{I}z\mathcal{R}z'$, $\mathcal{R}z$ and $\mathcal{I}z$ being the real and imaginary parts of z respectively. Note that if $\psi \in \mathcal{H}$ then $\psi \in \mathcal{H}_0$ if and only if $\psi(z) = f(z)e^{-\kappa|z|^2}$ where $f(z)$ is entire. Using (2.4) we can write the Hamiltonian in the form

$$H = \sum_{n\in\mathbb{Z}[i]} \omega_n f_{\zeta_n} \otimes \overline{f_{\zeta_n}},$$

where for $\zeta \in \mathbb{C}$,

$$f_\zeta(z) = \sqrt{\frac{\pi}{2\kappa}} P_0(z,\zeta) = \sqrt{\frac{2\kappa}{\pi}} \exp[2\kappa\bar{\zeta}z - \kappa(|\zeta|^2 + |z|^2)]. \tag{2.5}$$

Note that $\|f_\zeta\| = 1$, $\langle f_\zeta\,,\ f_{\zeta'}\rangle = \sqrt{\frac{\pi}{2\kappa}} f_{\zeta'}(\zeta)$ and that H is an integral operator with kernel

$$H(z,z') = \sum_{n\in\mathbb{Z}[i]} \omega_n f_{\zeta_n}(z)\overline{f_{\zeta_n}(z')}. \tag{2.6}$$

We first obtain a bound on $H(z,z')$ which implies that H is bounded. We give the following simple estimate without proof.

Lemma 2.1: *For* $s,\ t > 0$ *and* $z,\ z' \in \mathbb{C}$

$$\sum_{n \in \mathbb{Z}[i]} e^{-s|z-\zeta_n|^2} e^{-t|\zeta_n - z'|^2} \leq K(s+t) e^{-\frac{st}{s+t}|z-z'|^2}.$$

where

$$K(s) = 9 + 8e^{-s} + 4\left(\frac{\pi}{s}\right)^{\frac{1}{2}} + \frac{4}{s}.$$

The above Lemma implies that $|H(z,z')|$ is bounded above by

$$\frac{2R\kappa}{\pi} K(2\kappa) e^{-\frac{\kappa}{2}|z-z'|^2}, \tag{2.7}$$

where $R = \max(|a|, |b|)$. Therefore H is bounded and

$$\|H\| \leq 4RK(2\kappa). \tag{2.8}$$

Note that the heat kernel is

$$P_t(z,z') = \frac{1}{2\pi t} e^{-\frac{1}{2t}|z-z'|^2}$$

and the corresponding operator has unit norm. From now on we take κ sufficiently large so that $K(2\kappa) < 10$ and we let $\bar{R} = 40R$ so that $\|H\| \leq \bar{R}$.

III. The Spectrum of H.

Let $\{U_z : z \in \mathbb{C}\}$ be the family of unitary operators on $\mathcal{H}$ corresponding to the magnetic translations:

$$(U_z f)(z') = e^{2i\kappa z \wedge z'} f(z + z').$$

Then for $m \in \mathbb{Z}[i]$

$$U_m H(\omega, \zeta) U_m^{-1} = H(\tau_m(\omega, \zeta)). \tag{3.1}$$

Note that $[P_0, U_z] = 0$ for all $z \in \mathbb{C}$ so that $U_z \mathcal{H}_0 \subset \mathcal{H}_0$. Also $U_{z_1} U_{z_2} = e^{2i\kappa z_2 \wedge z_1} U_{z_1 + z_2}$. The ergodicity of $\{\tau_m : m \in \mathbb{Z}[i]\}$ and equation (3.1) together imply that the spectrum of $H(\omega, \zeta)$ and its components are non random (see Ref. 14 Th V.2.4).

Lemma 3.1: *With probability one*

$$[4a, 4b] \subset \sigma\left(H(\omega, \zeta)\right).$$

Proof: It is sufficient to prove that for each $E \in [4a, 4b]$ and for all $\delta > 0$, there exists $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') > 0$ and $\psi \in \mathcal{H}_0$ with $||\psi|| = 1$ such that for all $(\omega, \zeta) \in \Omega'$, $||\left(H(\omega,\ \zeta) - E\right)\psi|| < \delta$.

Let $B = \{0, 1, i, 1+i\}$. Choose $E \in [4a, 4b]$ and D such that $\sum_{n:|\zeta_n| \geq D} e^{-\kappa|\zeta_n - \zeta_0|^2} < \delta/4R$, where $R = \max(|a|, |b|)$, and let

$$\Omega_2' = \{\zeta \in \Omega_2 : |\zeta_n - \zeta_0| \leq \frac{\delta}{4E\sqrt{2\kappa}}\ \forall n \in B\}$$

then since for all $n \in B$, the impurities ζ_n and ζ_0 can be arbitrarily close to one another, $\mathbb{P}(\Omega_2') > 0$. Let

$$\Omega_1' = \{\omega \in \Omega_1 : |\omega_n - \frac{E}{4}| < \frac{\delta}{16}\ \forall n \in B \text{ and } \max_{m \notin B : |\zeta_m| < D} |\omega_m| K(\kappa) < \frac{\delta}{4}\}.$$

Since $E/4$ and 0 are in the support of μ, $\mathbb{P}(\Omega_1') > 0$. Let $\Omega' = \Omega_1' \times \Omega_2'$, then $\mathbb{P}(\Omega') > 0$.

Now

$$\begin{aligned}(Hf_{\zeta_0} - Ef_{\zeta_0})(z) &= \sum_{n \in B} (\omega_n - \frac{E}{4}) \langle f_{\zeta_n}, f_{\zeta_0} \rangle f_{\zeta_n}(z) + E\left(\frac{1}{4}\sum_{n \in B} \langle f_{\zeta_n}, f_{\zeta_0}\rangle f_{\zeta_n}(z) - f_{\zeta_0}(z)\right) \\ &+ \sum_{n \notin B : |\zeta_n| < D} \omega_n \langle f_{\zeta_n}, f_{\zeta_0}\rangle f_{\zeta_n}(z) + \sum_{n : |\zeta_n| \geq D} \omega_n \langle f_{\zeta_n}, f_{\zeta_0}\rangle f_{\zeta_n}(z).\end{aligned}$$

Hence

$$\begin{aligned}\|Hf_{\zeta_0} - Ef_{\zeta_0}\| &\leq \frac{\delta}{4} + \frac{E}{4}\| \sum_{n\in B} \langle f_{\zeta_n}, f_{\zeta_0}\rangle f_{\zeta_n} - 4f_{\zeta_0}\| \\ &+ \sum_{n\notin B\,:|\zeta_n|<D} |\omega_n|\,|\langle f_{\zeta_n}, f_{\zeta_0}\rangle| + R \sum_{n:|\zeta_n|\geq D} e^{-\kappa|\zeta_n-\zeta_0|^2}.\end{aligned}$$

It is easily seen that $\|\langle f_{\zeta_n}, f_{\zeta_0}\rangle f_{\zeta_n} - f_{\zeta_0}\|^2 = 1 - |\langle f_{\zeta_n}, f_{\zeta_0}\rangle|^2$ is bounded by $2\kappa|\zeta_n - \zeta_0|^2$, and therefore for all $(\omega, \zeta) \in \Omega'$,

$$\|Hf_{\zeta_0} - Ef_{\zeta_0}\| < \delta.$$

□

We now state the main theorem, which we will prove in the sequel.

Theorem 3.2 *There exists $\kappa_0 > 0$ such that for $\kappa > \kappa_0$, with probability one,*

(a) 0 is an eigenvalue of H with infinite multiplicity,

(b) $\sigma_{\text{cont}}\,(H) = \emptyset$,

(c) if $\lambda \in \sigma(H)\backslash\{0\}$, is an eigenvalue of H with eigenfunction ψ, then ψ decays exponentially with rate $\geq \kappa^{1/4}$.

We will start by showing part (a). The lemma is very close to results proven in Refs 8 and 13 so only a sketch of the proof will be given.

Lemma 3.3: *There exists $\kappa_1 > 0$ such that for $\kappa > \kappa_1$, with probability one, 0 is an eigenvalue of H with infinite multiplicity.*

Proof: Let

$$\psi_\zeta(z) = \prod_{n\in\mathbb{Z}[i]} (1 - \frac{z}{\zeta_n})e^{\frac{z}{\zeta_n} + \frac{z^2}{2\zeta_n^2}}.$$

If we can show that the sums $\sum_n 1/|\zeta_n|^3$ and $\sum_n 1/\zeta_n^2$ converge independently of ζ then it follows from the theory of entire functions (see Ref. 15) that there exists $A > 0$ and $R > 0$, both independent of ζ such that for $|z| > R$, $|\psi_\zeta(z)| \le e^{A|z|^2}$. The first sum is easily bounded, the second can be bounded by utilising the four-fold rotational symmetry of the $\mathbb{Z}[i]$ to cancel any large contributions. Let $\phi_k(z) = z^k \psi_\zeta(z) e^{-\kappa|z|^2}$ for $k \ge 1$, then if $\kappa > A$, the ϕ_k's are in $\mathcal{H}_0$ and $\phi_k(\zeta_n) = 0$ for all $n \in \mathbb{Z}[i]$. Therefore $H\phi_k = 0$ for all $k \ge 1$. Moreover if $\sum_{j=1}^N a_{k_j}\phi_{k_j} = 0$ then $\sum_{j=1}^N a_{k_j} z^{k_j} = 0$ for $z \notin \{\zeta_n\}$. Therefore $\sum_{j=1}^N a_{k_j} z^{k_j} \equiv 0$ and thus the a_{k_j}'s are zero implying that the ϕ_k's are independent.

$\square$

For Theorem 3.2 parts (b), (c) we can simplify the problem by studying the behaviour of the generalized eigenfunctions at the impurity sites only. We have

$$(H\psi)(z) = \frac{\pi}{2\kappa} \sum_{n \in \mathbb{Z}[i]} \omega_n P_0(z, \zeta_n) \psi(\zeta_n)$$

and thus if $(H\psi)(z) = \lambda\psi(z)$,

$$\frac{\pi}{2\kappa} \sum_{n \in \mathbb{Z}[i]} \omega_n P_0(z, \zeta_n) \psi(\zeta_n) = \lambda \psi(z) \tag{3.2}$$

which evaluated at ζ_m gives

$$\frac{\pi}{2\kappa} \sum_{n \in \mathbb{Z}[i]} \omega_n P_0(\zeta_m, \zeta_n) \psi(\zeta_n) = \lambda \psi(\zeta_m)$$

Let $\omega_n \psi(\zeta_n) = \xi_n$. Then

$$\frac{\pi}{2\kappa} \sum_{n \in \mathbb{Z}[i]} P_0(\zeta_m, \zeta_n) \xi_n = \frac{\lambda}{\omega_m} \xi_m \tag{3.3}$$

We can thus reduce the problem to the study of a random matrix which has ω-dependent elements on the diagonal and ζ-dependent rapidly decaying off-diagonal elements. We

write this matrix as a sum of a diagonal matrix and an off-diagonal matrix as defined below.

Let $\mathcal{M} = l^2(\mathbb{Z}[i])$. Define the operators M_0, and V_ω^λ on $\mathcal{M}$ as follows.

$$\begin{aligned} \langle m|M_0|n\rangle &= \frac{\pi}{2\kappa}P_0(\zeta_m,\zeta_n)(1-\delta_{mn}) \\ \langle m|V_\omega^\lambda|n\rangle &= \left(1-\frac{\lambda}{\omega_n}\right)\delta_{mn}. \end{aligned} \tag{3.4}$$

For a proof of Theorem 3.2 part (c) we note that the eigenvectors ξ of $M^\lambda = M_0 + V_\omega^\lambda$, are related by an explicit formula to the generalized eigenfunctions ψ of H in such a way that exponential decay of the former implies exponential decay of the latter:

¿From (3.2), if $\lambda \neq 0$

$$\psi(z) = \frac{\pi}{2\kappa\lambda}\sum_{n\in\mathbb{Z}[i]} P_0(z,\zeta_n)\xi_n$$

If ξ_n decays exponentially, $|\xi_n| \leq Ce^{-m|\zeta_n|}$ we have

$$\begin{aligned} |\psi(z)| &\leq \frac{C}{\lambda}\sum_{n\in\mathbb{Z}[i]} e^{-\kappa|z-\zeta_n|^2}e^{-m|\zeta_n|} \leq \frac{C}{\lambda}e^{-m|z|}\sum_{n\in\mathbb{Z}[i]} e^{-\kappa|z-\zeta_n|^2}e^{m|z-\zeta_n|} \\ &\leq \frac{C}{\lambda}e^{-m|z|}e^{\frac{m^2}{2\kappa}}\sum_{n\in\mathbb{Z}[i]} e^{-\frac{\kappa}{2}|z-\zeta_n|^2} \leq \frac{C}{\lambda}e^{\frac{m^2}{2\kappa}}K(\frac{\kappa}{2})e^{-m|z|} \end{aligned} \tag{3.5}$$

and $\psi(z)$ decays exponentially, where we have bounded the sum by taking $s = \kappa/2$, $t = 0$ in Lemma 2.1.

Thus we want to show that the eigenvectors for the eigenvalue equation $M^\lambda\xi = 0$ decay exponentially for $\lambda \neq 0$. We will do this by the same method as in Ref. 8. First we will need to make a few definitions.

For regions $\Lambda \subset \mathbb{Z}[i]$ we define M_Λ^λ to be the restriction of M^λ to $l^2(\Lambda)$. If $E \notin \sigma(M_\Lambda^\lambda)$ then the Green function

$$\Gamma_\Lambda^\lambda(E) = (M_\Lambda^\lambda - E)^{-1} \tag{3.6}$$

is well-defined. In particular, we shall consider the regions

$$\Lambda_L(n) = \{n' \in \mathbb{Z}[i]| \ : \ |n' - n|_\infty < L/2\} \tag{3.7}$$

for $n \in \mathbb{Z}[i]$ and $L > 0$.

Definition. *Fix constants $\beta \in (0,1)$ and $s \in (\frac{1}{2}, 1)$. Given a configuration (ω, ζ), a square $\Lambda_L(n)$ is called (m, E)-regular for some $m > 0$ and $E \in \mathbb{R}$ if the following two conditions are satisfied:*

(RA)
$$d(E, \sigma(M^\lambda_{\Lambda_L(n)}(\omega, \zeta))) > \frac{1}{2} e^{-L^\beta},$$

(RB)
$$|\langle n|\Gamma^\lambda_{\Lambda_L(n)}(E)|n'\rangle| \leq e^{-mL}$$

for all $n' \in \tilde{\Lambda}_L(n)$ where $\tilde{\Lambda}_L(n) = \Lambda_L(n) \setminus \Lambda_{\tilde{L}}(n)$ with $\tilde{L} = L - L^s$. $\Lambda_L(n)$ is called singular if it is not regular.

We now state a theorem which is an variant of the main theorem in Ref. 2 where the von Dreifus and Klein Theorem is adapted from the case of a tight-binding (finite range) Hamiltonian to the case where the Hamiltonian has a long range hopping term with Gaussian decay. It states conditions under which the eigenvectors of the random matrix M^λ with eigenvalue 0 decay exponentially.

Theorem 3.4 *Fix constants $\beta \in (0,1)$, $s \in (\frac{1}{2}, 1)$, $\gamma \in (0,1)$, $p > 2$, $q > 4p + 12$. There exists $Q_0 > 0$ depending on all these constants but* **independent of λ and** *$\kappa > 1$ such that the following holds: If for λ, κ the conditions* (P1) *and* (P2) *are satisfied, where*

(P1) *There exists an $L_0 > Q_0$ and m_0 such that*

$$\mathbb{P}\left\{\Lambda_{L_0}(0) \textit{ is } (m_0, 0)\textit{-regular}\right\} \geq 1 - L_0^{-p} \tag{3.8}$$

(P2) *There exists $\eta > 0$ such that, for all $E \in (-\eta, \eta)$ and for all $L > L_0$,*

$$\mathbb{P}\left\{d\left(E, \sigma\left(M^{\lambda}_{\Lambda_L(0)}\right)\right) < e^{-L^{\beta}}\right\} < L^{-q}, \tag{3.9}$$

then, for all $m \in (0, m_0)$, there exists $\delta > 0$ depending on m, m_0, L_0, β and η such that for all $E \in (-\delta, \delta)$ the eigenvectors of M^{λ} with eigenvalue E decay exponentially with rate $\geq m$.

The main work of this paper consists in proving that the conditions (P1) and (P2) are satisfied. The conditions can be seen to consist of two types of estimate. (RB) of (P1) is an estimate of the decay of the Greens function $\Gamma^{\lambda}_{\Lambda_L}(0)$, while (RA) of (P1) and (P2) are Wegner type estimates that require small gaps in the Λ_L dependent spectrum. It is unusual that it is the latter estimates that will require the finer analysis; previous works have found the decay of the Green's function to require the more delicate study. This is because we want to show that the eigenfunctions are exponentially decaying for arbitrarily small λ. Inspecting (3.4) we see that for λ small the ω-dependence becomes less significant and does not give sufficient randomness for Wegner type estimates. Therefore we have to utilise randomness provided by the positions of the ζ_j's.

IV. Proof of the Conditions (P1) and (P2)

We will begin by showing (RB) of (P1). We will need the following two probabilistic lemmas in which we fix $u > 3$.

Lemma 4.1: *There exists Q_1 such that*

$$\mathbb{P}\left(|\zeta_n - \zeta_{n'}| > \frac{2}{L^u} \text{ for all } n, n' \in \Lambda_L,\ n \neq n'\right) \geq 1 - \frac{1}{L^{u-3}} \tag{4.1}$$

for all $L > Q_1$.

Proof: The ζ_n's have a distribution with a density bounded by r_b for each B_n and thus,

$$\mathbb{P}\left(|\zeta_n - \zeta_{n'}| > \frac{2}{L^u} \text{ for all } n, n' \in \Lambda_L,\ n \neq n'\right) \geq \left(1 - \frac{4r_b}{L^u}\right)^{L^2}.$$

By taking L sufficiently large we get the result.

$\square$

Lemma 4.2: *There exists Q_2 such that*

$$\mathbb{P}\left(\left|1 - \frac{\lambda}{\omega_n}\right| > \frac{1}{L^u} \quad \forall\, n \in \Lambda_L\right) > 1 - \frac{1}{L^{u-3}} \tag{4.2}$$

for all $L > Q_2$.

Proof: Now $\left|1 - \frac{\lambda}{\omega_n}\right| \leq \frac{1}{L^u}$ gives us that $-\frac{1}{L^u} \leq 1 - \frac{\lambda}{\omega_n} \leq \frac{1}{L^u}$ which implies that $1 + \frac{1}{L^u} \geq \frac{\lambda}{\omega_n} \geq 1 - \frac{1}{L^u}$.

Thus ω_n must fall between the bounds,

$$\frac{|\lambda|}{1 + \frac{1}{L^u}} \leq\ |\omega_n|\ \leq \frac{|\lambda|}{1 - \frac{1}{L^u}}. \tag{4.3}$$

Hence

$$\begin{aligned}
\mathbb{P}\left(\left|1 - \frac{\lambda}{\omega_n}\right| \leq \frac{1}{L^u}\right) &\leq 2\rho_b|\lambda|\left(\frac{1}{1 - \frac{1}{L^u}} - \frac{1}{1 + \frac{1}{L^u}}\right) \\
&= \frac{4\rho_b|\lambda|}{L^u}(1 - \frac{1}{L^{2u}})^{-1} \leq \frac{2^{u+2}\rho_b\bar{R}}{L^u}.
\end{aligned}$$

if $L > 2$, where we have used $1 - 1/L^{2u} > 1/2^u$. Therefore

$$\mathbb{P}\left(\left|1 - \frac{\lambda}{\omega_n}\right| > \frac{1}{L^u} \quad \forall\, n \in \Lambda_L\right) > \left(1 - \frac{2^{u+2}\rho_b\bar{R}}{L^u}\right)^{(L+1)^2}.$$

By taking L sufficiently large we get the result.

$\square$

The following Lemma is proved in Ref. 8.

Lemma 4.3: *For all $\gamma \in (0,1)$, there exists $C_0(\gamma) > 0$ such that for $\alpha > 1$*

$$\sum_{m \in \mathbb{Z}[i]} e^{-\alpha\{|z-m|^\gamma + |z'-m|^\gamma\}} \leq C_0(\gamma) e^{-\alpha|z-z'|^\gamma}. \tag{4.4}$$

The following Lemma will be used to show (RB) of (P1).

Lemma 4.4: *For all $\gamma \in (0,1)$ and $u > 3$, there exists Q_3 such that for all $L > Q_3$ and all $\kappa > L^{4u}/4$ we have for any $n, n' \in \Lambda_L$,*

$$\mathbb{P}\left(|\langle n|\Gamma^\lambda_{\Lambda_L}(0)|n'\rangle| \leq 2L^u e^{-\frac{\kappa^{1/2}}{8}|n-n'|^\gamma} \right) > 1 - \frac{2}{L^{u-3}}. \tag{4.5}$$

Proof: In the following we let $\gamma \in (0,1)$. Using $|a+b|^2 \leq 2(|a|^2 + |b|^2)$ we have $\kappa|\zeta_n - \zeta_{n'}|^2 \geq \kappa(\frac{1}{2}|n-n'|^2 - 2) > \frac{\kappa}{4}|n-n'|^2 > \frac{\kappa^{1/2}}{4}|n-n'|^\gamma$ for $|n-n'| \geq 3$. Suppose that $|\zeta_n - \zeta_{n'}| > 2/L^u$ for all $n, n' \in \Lambda_L$, $n \neq n'$. Then for $|n-n'| < 3$ we have that $\kappa|\zeta_n - \zeta_{n'}|^2 > \frac{\kappa^{1/2}}{4}|n-n'|^\gamma$ if $\frac{2\kappa^{1/2}}{L^{2u}} > 1$. Thus we can write

$$e^{-\kappa|\zeta_n - \zeta_{n'}|^2} \leq e^{-\frac{\kappa^{1/2}}{4}|n-n'|^\gamma}$$

and consequently

$$|\langle n|M^0_{\Lambda_L}|n'\rangle| \leq e^{-\frac{\kappa^{1/2}}{4}|n-n'|^\gamma}(1-\delta_{nn'}).$$

If $|1-\lambda/\omega_n| > 1/L^u$ for all $n \in \Lambda_L$ then for all $n, n' \in \Lambda_L$ we also have,

$$|\langle n|(V^\lambda_{\Lambda_L})^{-1}|n'\rangle| \leq L^u \delta_{nn'}.$$

Therefore we can write

$$|\langle n|(V^\lambda_\Lambda)^{-1} M^0_{\Lambda_L}|n'\rangle| \qquad \leq \sum_{p \in \Lambda} |\langle n|(V^\lambda_\Lambda)^{-1}|p\rangle||\langle p|M^0_{\Lambda_L}|n'\rangle|$$

$$
\begin{aligned}
&= |\langle n|(V_\Lambda^\lambda)^{-1}|n\rangle||\langle n|M^0_{\Lambda_L}|n'\rangle| \\
&\le L^u e^{-\frac{\kappa^{1/2}}{8}} e^{-\frac{\kappa^{1/2}}{8}|n-n'|^\gamma}(1-\delta_{nn'}),
\end{aligned}
$$

and

$$
\begin{aligned}
|\langle n|\left((V_{\Lambda_L}^\lambda)^{-1}M^0_{\Lambda_L}\right)^2|n'\rangle| &\le \sum_{r\in\Lambda} |\langle n|(V_{\Lambda_L}^\lambda)^{-1}|n\rangle||\langle n|M^0_{\Lambda_L}|r\rangle||\langle r|(V_{\Lambda_L}^\lambda)^{-1}|r\rangle||\langle r|M^0_{\Lambda_L}|n'\rangle| \\
&\le L^{2u}e^{-\frac{\kappa^{1/2}}{4}} \sum_{r\in\mathbb{Z}[i]} e^{-\frac{\kappa^{1/2}}{8}|n-r|^\gamma} e^{-\frac{\kappa^{1/2}}{8}|r-n'|^\gamma}(1-\delta_{nr})(1-\delta_{rn'}) \\
&\le C_0(\gamma)L^{2u}e^{-\frac{\kappa^{1/2}}{4}} e^{-\frac{\kappa^{1/2}}{8}|n-n'|^\gamma}.
\end{aligned}
$$

Similarly,

$$
|\langle n|\left((V_{\Lambda_L}^\lambda)^{-1}M^0_{\Lambda_L}\right)^k|n'\rangle| \le C_0(\gamma)^{k-1}L^{ku}e^{-\frac{k\kappa}{8}^{1/2}} e^{-\frac{\kappa^{1/2}}{8}|n-n'|^\gamma}.
$$

Let T be the operator with $\langle n|T|m\rangle = e^{-\frac{\kappa^{1/2}}{8}|n-m|^\gamma}$. Then we can make $\| \left((V_{\Lambda_L}^\lambda)^{-1}M^0_{\Lambda_L}\right)^k \| \le \frac{1}{2^k}\|T\|$ by making $C_0(\gamma)L^u e^{-\frac{\kappa^{1/2}}{8}} < \frac{1}{2}$. We can therefore iterate the resolvent identity to get,

$$
\begin{aligned}
\Gamma^\lambda(0) &= (V^\lambda)^{-1} - (V^\lambda)^{-1}M^0\Gamma^\lambda(0) \\
&= \sum_{k=0}^{\infty}(-1)^k\left((V^\lambda)^{-1}M^0\right)^k (V^\lambda)^{-1}
\end{aligned}
$$

Hence, if we take $L > Q_3$ with Q_3 larger than Q_1 and Q_2 and sufficiently large that $\frac{1}{2}L^{2u} > 8\ln(2C_0(\gamma)L^u)$, the result follows from Lemmas 4.1 and 4.2.

$\square$

Let β be fixed as in Theorem 3.4 and $\kappa > \pi/2$. To prove (RA) of (P1) and condition (P2) we need to look at two regimes, $|\lambda| \ge e^{-L^\beta/2}$ and $|\lambda| < e^{-L^\beta/2}$. The next lemma deals with the first regime, and the Lemmas 4.6 - 4.9 with the second.

Lemma 4.5: *If* $|\lambda| \geq e^{-L^\beta/2}$, $L > 1$, $E \in \mathbb{R}$ *and* $\epsilon > 0$, *then*

$$\mathbb{P}(d(E, \sigma(M^\lambda_{\Lambda_L})) < \epsilon) < 8\rho_b R^2 L^2 e^{L^\beta/2}\epsilon. \tag{4.6}$$

Proof: First we need to find a bound on the density of the diagonal terms of M^λ.

$$\begin{aligned}
\sup_x \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_{x-\epsilon<1-\frac{\lambda}{\omega}<x+\epsilon} \rho(\omega)d\omega &= \sup_x \lim_{\epsilon\to 0} \frac{1}{2\epsilon\lambda}\int_{x-\epsilon}^{x+\epsilon} \rho\left(\frac{\lambda}{1-u}\right)\left(\frac{\lambda}{1-u}\right)^2 du \\
&< \sup_x \lim_{\epsilon\to 0} \frac{\rho_b R^2}{2\epsilon|\lambda|}\int_{x-\epsilon}^{x+\epsilon} du = \frac{\rho_b R^2}{|\lambda|}.
\end{aligned} \tag{4.7}$$

It follows that the density of $x_{nn} = \langle n|M^\lambda_\Lambda|n\rangle$ is bounded by $\rho_b R^2 e^{L^\beta/2}$.

For Borel subsets B of $\mathbb{R}$ let $\sigma^\Lambda_n(B) = \langle n|E_\Lambda(B)|n\rangle$, where $E_\Lambda(B)$ are the spectral projections of M^λ_Λ. Then by Lemma VIII.1.8 in Ref. 14, and by (4.7)

$$\mathbb{E}_{x_{nn}}\sigma^\Lambda_n(B) < \rho_b R^2 e^{L^\beta/2}\int_B dx$$

and therefore

$$\mathbb{E}\sigma^\Lambda_n(B) < \rho_b R^2 e^{L^\beta/2}\int_B dx.$$

As in Proposition VIII.4.11 of Ref. 14, it then follows that, using (4.7), for all $E \in \mathbb{R}$ and $\epsilon > 0$,

$$\mathbb{P}(d(E, \sigma(M^\lambda_{\Lambda_L})) < \epsilon) < 2\rho_b R^2 e^{L^\beta/2}\epsilon|\Lambda_L| < 2\rho_b R^2 e^{L^\beta/2}\epsilon(L+1)^2.$$

If $L \geq 1$ the result follows.

$\square$

For the next part it is necessary to make some definitions.

Definitions: We define $V = \{\zeta_n\,, n \in \mathbb{Z}[i]\}$ to be the vertices of a graph $\mathcal{G}(V, E)$ with edges E defined as $E = \{(\zeta_m, \zeta_n) \;:\; |\zeta_m - \zeta_n| < 1/8,\ \forall\ n, m \in \mathbb{Z}[i], n \neq m\}$. The *degree*

of a vertex, $\deg(\zeta_m) = \#\{n \in \mathbb{Z}[i] : (\zeta_m, \zeta_n) \in E\}$. Two vertices are said to be *connected* if there exists a path between them along a series of edges. A *component* is defined to be a maximally connected subgraph. We will define a *cluster* to be a component of the graph $\mathcal{G}(V, E)$.

Lemma 4.6: *For each configuration $\{\zeta_n\}$ there exist clusters containing at most 4 vertices such that the distance between every pair of clusters is at least* $1/8$.

Proof: The distance between two clusters $\mathcal{C}_i$, $\mathcal{C}_j$ is given by,

$$d(\mathcal{C}_i, \mathcal{C}_j) = \inf\{\, d(\zeta_n^i, \zeta_m^j) \,|\, \zeta_n^i \in \mathcal{C}_i,\ \zeta_m^j \in \mathcal{C}_j\}.$$

It is easily seen that if the distance between two clusters is less than 1/8, then the distance between one of the vertices in one cluster, must be closer than 1/8 to a vertex in the other. Thus an edge will exist that connects the two clusters, leading to a contradiction in their definition as separate clusters.

It suffices to show that we cannot have a cluster with more than four vertices. The diameter of a cluster is given by,

$$\operatorname{diam}(\mathcal{C}_i) = \sup\{\, d(\zeta_n, \zeta_m) \mid \zeta_n, \zeta_m \in \mathcal{C}_i\}.$$

We know that the unit squares centred on the Gaussian integers, $\{B_n, n \in \mathbb{Z}[i]\}$, contain exactly one vertex each. The maximum diameter for a cluster of five vertices will be less than 1/2 by virtue of the definition of a cluster. However a circle of diameter 1/2 cannot intersect more than four of the B_n, so we cannot have a cluster of five. If we had a cluster of more than five vertices, we could also have a cluster of five as can be seen if we perform a one by one deletion of lowest degree vertices until only five remain. Thus we cannot

have a cluster with more than five vertices.

$\square$

For a configuration $\{\zeta_n\}$, let

$$\langle n|\tilde{M}_\Lambda^\lambda|n'\rangle = \begin{cases} \langle n|M_\Lambda^\lambda|n'\rangle & \text{if } \zeta_n, \zeta_{n'} \text{ are in the same cluster,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_N$ be the clusters in Λ and let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_N$ be the projections onto $\mathcal{H}_i$ the space spanned by $\{|n\rangle \,:\, \zeta_n \in \mathcal{C}_i\}$. Let

$$M_i = \mathcal{P}_i M_\Lambda^\lambda \mathcal{P}_i = \mathcal{P}_i \tilde{M}_\Lambda^\lambda \mathcal{P}_i. \tag{4.8}$$

Lemma 4.7: *For* $\lambda = 0$,

$$|\det M_i| \geq \quad C \prod_{m< n:\, \zeta_m,\, \zeta_n \in \mathcal{C}_i} \left(1 - e^{-\kappa|\zeta_m - \zeta_n|^2}\right), \tag{4.9}$$

where $C > 0$ *is a constant independent of* ζ.

Note that numerical calculation shows that the inequality is satisfied with $C = 1$. If the ζ_i's are distinct then for $\lambda = 0$ we can write

$$\begin{aligned} \langle \xi, M_i \xi \rangle &= \frac{\pi}{2\kappa} \sum_{m,n:\zeta_m,\zeta_n \in \mathcal{C}_i} \bar{\xi}_m P_0(\zeta_m, \zeta_n) \xi_n = \frac{\pi}{2\kappa} \int_{\mathbb{C}} \sum_{m,n:\zeta_m,\zeta_n \in \mathcal{C}_i} \bar{\xi}_m P_0(\zeta_m, z) P_0(z, \zeta_n) \xi_n \; dz \\ &= \int_{\mathbb{C}} |\sum_{m:\zeta_m \in \mathcal{C}_i} \xi_m f_{\zeta_m}(z)|^2 dz \; > \; 0 \end{aligned} \tag{4.10}$$

since the f_{ζ_i}'s will be linearly independent. Thus $\det M_i > 0$.

Proof: If $\lambda = 0$, recall that from the definition of M^λ, for ζ_m, ζ_n in a cluster $\mathcal{C}_i$,

$$\langle m|M_i|n\rangle = e^{-\kappa|\zeta_m - \zeta_n|^2 - 2i\kappa\zeta_m \wedge \zeta_n}.$$

We only have to prove the result up to a cluster of four. For a cluster of one, the result is trivial. For a cluster of two we get,

$$|\det M_i| = 1 - e^{-2\kappa|\zeta_1-\zeta_2|^2}.$$

We now give the proof for a cluster of three. A direct proof with $C = 1$ can be given (see Ref 16) but it is difficult to extend this to the case of a cluster of four impurities. For this reason we shall give an indirect proof which can be extended to the latter case.

Let $\kappa^{\frac{1}{2}}(\zeta_2 - \zeta_1) = ae^{i\alpha}$ and $\kappa^{\frac{1}{2}}(\zeta_3 - \zeta_1) = be^{i\beta}$. Then $\det M_i = G_3(a, b, \phi)$ where $\phi = \alpha - \beta$ and

$$G_3(a, b, \phi) = 1 - e^{-2a^2} - e^{-2b^2} - e^{-2c^2} + 2e^{-(a^2+b^2+c^2)} \cos(2ab \sin(\phi)), \tag{4.11}$$

with

$$c^2 = a^2 + b^2 - 2ab \cos \phi.$$

Note that without loss of generality we can take $\phi \in [0, \pi]$. G_3 is an analytic function of a, b and ϕ. It is easy to check that,

$$G_3(0, b, \phi) = G_3(a, 0, \phi) = G_3(a, ae^{\pm i\phi}, \phi) = 0,$$

and

$$\frac{\partial G_3}{\partial a}(0, b, \phi) = \frac{\partial G_3}{\partial b}(a, 0, \phi) = 0$$

so that we can write

$$G_3(a, b, \phi) = a^2b^2(b - ae^{-i\phi})(b - ae^{i\phi})g_3(a, b, \phi) = a^2b^2c^2g_3(a, b, \phi) \tag{4.12}$$

where $g_3(a, b, \phi)$ is an analytic function of a, b and ϕ.

Let $A = \dot{\mathbb{R}}_+^2 \times [0, \pi]$, where $\dot{\mathbb{R}}_+$ denotes the one-point compactification of $\mathbb{R}_+$, and let A_o be the interior of A. Define $f_3 : A_o \to \mathbb{R}$ by

$$f_3(a, b, \phi) = \frac{G_3(a, b, \phi)}{(1 - e^{-a^2})(1 - e^{-b^2})(1 - e^{-c^2})}. \tag{4.13}$$

$f_3(a, b, \phi) > 0$ for all $(a, b, \phi) \in A_o$ by the inequality (4.10). Note that $c = 0$ only if $a = b = 0$ or $a = b$ and $\phi = 0$. We shall prove that for each point (a_0, b_0, ϕ_0) on the boundary of A, we have $\liminf_{(a,b,\phi)\to(a_0,b_0,\phi_0)} f_3(a, b, \phi) > 0$. Then since A is compact there exists $C > 0$ such that $f_3(a, b, \phi) > C$ for all $(a, b, \phi) \in A$.

For points on the boundary of A for which a, b and c are all finite and non-zero $f_3(a, b, \phi)$ is defined by (4.13) and is strictly positive. Now $\lim_{a\to\infty} f_3(a, b, \phi) = 1 + e^{-b^2} > 1$ for all $(b, \phi) \in \mathbb{R}_+ \times [0, \pi]$ and similarly for $\lim_{b\to\infty} f_3(a, b, \phi)$. Also $\liminf_{(a,b)\to(\infty,\infty)} f_3(a, b, \phi) = 1$.

Next we have that $\lim_{(a,b)\to(0,0)} f_3(a, b, \phi) = g_3(0, 0, \phi)$ and we can check that $g_3(0, 0, \phi) = 4$. For $b > 0$, $\lim_{a\to0} f_3(a, b, \phi) = b^4 g_3(0, b, \phi)/(1 - e^{-b^2})^2$. We can calculate $g_3(0, b, \phi)$ explicitly to get $b^4 g_3(0, b, \phi) = 2e^{-2b^2}(e^{2b^2} - 1 - 2b^2) > 0$. Similarly we can show that $g_3(a, 0, \phi) > 0$. Finally, by symmetry it follows that $\lim_{a\to b,\ \phi\to0} f_3(a, b, \phi) = \lim_{a\to0} f_3(a, b, \psi)$ where ψ is the angle between the edges of lengths b and c, which has already been shown to be strictly positive. Note that in fact this limit is independent of ψ.

Now we come to the proof of the Lemma for a cluster of four. The idea of the proof is the same as for a cluster of three but the details are more complicated.

Let $\kappa^{\frac{1}{2}}(\zeta_2 - \zeta_1) = ae^{i\alpha}$, $\kappa^{\frac{1}{2}}(\zeta_3 - \zeta_1) = be^{i\beta}$ and $\kappa^{\frac{1}{2}}(\zeta_4 - \zeta_1) = ce^{i\gamma}$. Then $\det M_i = G_4(a, b, c, \phi, \psi)$ where $\phi = \beta - \alpha$, $\psi = \alpha - \gamma$ and

$$G_4(a, b, c, \phi, \psi) = 1 - e^{-2a^2} - e^{-2b^2} - e^{-2c^2} - e^{-2u^2} - e^{-2v^2} - e^{-2w^2}$$

$$+e^{-2(b^2+v^2)} + e^{-2(a^2+w^2)} + e^{-2(c^2+u^2)}$$

$$+2e^{-(a^2+b^2+u^2)}\cos(4\Delta_{abu}) + 2e^{-(b^2+c^2+w^2)}\cos(4\Delta_{bcw})$$

$$+2e^{-(a^2+c^2+v^2)}\cos(4\Delta_{acv}) + 2e^{-(u^2+v^2+w^2)}\cos(4\Delta_{uvw})$$

$$-2e^{-(b^2+c^2+u^2+v^2)}\cos(4(\Delta_{acv}+\Delta_{abu}))$$

$$-2e^{-(a^2+c^2+u^2+w^2)}\cos(4(\Delta_{abu}-\Delta_{bcw}))$$

$$-2e^{-(a^2+b^2+v^2+w^2)}\cos(4(\Delta_{acv}-\Delta_{bcw})) \tag{4.14}$$

with

$$u^2 = b^2 + a^2 - 2ba\cos\phi \ ,$$

$$v^2 = c^2 + a^2 - 2ac\cos\psi \ ,$$

$$w^2 = b^2 + c^2 - 2bc\cos(\phi+\psi) \ ,$$

$$\Delta_{abu} = \frac{1}{2}ba\sin\phi \ ,$$

$$\Delta_{acv} = \frac{1}{2}ac\sin\psi \ ,$$

$$\Delta_{bcw} = \frac{1}{2}bc\sin(\phi+\psi) \ ,$$

and

$$\Delta_{uvw} = \frac{1}{2}\left(ba\sin\phi + ac\sin\psi - bc\sin(\phi+\psi)\right) \ .$$

$G_4(a,b,c,\phi,\psi)$ is an analytic function of a, b, c, ϕ and ψ. In this case also we can check that,

$$G_4(0,b,c,\phi,\psi) = G_4(a,0,c,\phi,\psi) = G_4(a,b,0,\phi,\psi) = 0,$$

$$\frac{\partial G_4}{\partial a}(0,b,c,\phi,\psi) = \frac{\partial G_4}{\partial b}(a,0,c,\phi,\psi) = \frac{\partial G_4}{\partial c}(a,b,0,\phi,\psi) = 0,$$

and

$$G_4(be^{\pm i\phi}, b, c, \phi, \psi) = G_4(ce^{\pm i\psi}, b, c, \phi, \psi) = G_4(a, ce^{\pm i(\phi+\psi)}\, c, \phi, \psi) = 0.$$

These identities imply that

$$G_4(a, b, c, \phi, \psi) = a^2 b^2 c^2 u^2 v^2 w^2 g(a, b, c, \phi, \psi) \tag{4.15}$$

where $g_4(a, b, c, \phi, \psi)$ is an analytic function of a, b, c, ϕ and ψ.

In this case we let $A = \dot{\mathbb{R}}_+^3 \times [0, \pi]^2$ and define $f_4 : A_o \to \mathbb{R}$ by

$$f_4(a, b, c, \phi, \psi) = \frac{G_4(a, b, c, \phi, \psi)}{(1 - e^{-a^2})(1 - e^{-b^2})(1 - e^{-c^2})(1 - e^{-u^2})(1 - e^{-v^2})(1 - e^{-w^2})}. \tag{4.16}$$

Using the same arguments as before it is sufficient to check that for each point $z = (a_0, b_0, c_0, \phi_0, \psi_0)$ on the boundary of A, we have $\liminf_{(a,b,c,\phi,\psi)\to z} f_4(a, b, c, \phi, \psi) > 0$.

When one of a, b and c tend to ∞, the problem simplifies to the three impurity case and taking the lower limit when two of them tend to ∞, reduces the problem to the two impurity case. When all of a, b and c tend to ∞ the lower limit is equal to 1. It remains to show that $f_4(a, b, c, \phi, \psi)$ is strictly positive in the limit of any subset of $\{a, b, c\}$ going to zero. By symmetry we need only check the cases $a, b, c \to 0$, $a, b \to 0$ and $a \to 0$. Now $\lim_{(a,b,c)\to(0,0,0)} f_4(a, b, c, \phi, \psi) = g(0, 0, 0, \phi, \psi) = 16/3$. Similarly $\lim_{(a,b)\to(0,0)} f_4(a, b, c, \phi, \psi) = 4e^{-2c^2}(e^{2c^2} - 1 - 2c^2 - 2c^4)/(1 - e^{-c^2})^3 > 0$.

Finally we need to check that $\lim_{a\to 0} f_4(a, b, c, \phi, \psi) > 0$. This is considerably more difficult and will be checked over several stages. We have that

$$\lim_{a\to 0} f_4(a, b, c, \phi, \psi) = 2h(b, c, \theta)/(1 - e^{-b^2})^2(1 - e^{-c^2})^2(1 - e^{-w^2}) \tag{4.17}$$

where

$$h(b, c, \theta) = \quad 1 - (1 + 2b^2)e^{-2b^2} - (1 + 2c^2)e^{-2c^2} - e^{-2w^2}$$

$$+2w^2e^{-2(b^2+c^2)} + 2e^{-b^2+c^2+w^2}\cos(2bc\sin\theta)$$

$$+4bce^{-(b^2+c^2+w^2)}(\cos\theta\cos(2bc\sin\theta) + \sin\theta\sin(2bc\sin\theta)) \qquad (4.18)$$

and $\theta = \phi + \psi$. Now differentiating h with respect to θ gives

$$\frac{dh}{d\theta} = 8bce^{-(b^2+c^2+w^2)}S(bc,\theta) \qquad (4.19)$$

where $S(x,\theta) = \sin\theta\,(\cosh(2x\cos\theta) - \cos(2x\sin\theta)) - x\sin(2x\sin\theta)$.

We can use $\cosh t \geq 1 + t^2/2! + t^4/4!$ for all t and $\sin t \leq \sum_{n=0}^{4}(-1)^n t^{2n+1}/(2n+1)!$, $\cos t \leq \sum_{n=0}^{4}(-1)^n t^{2n}/(2n)!$ for $t < 10$ to write

$$S(x,\theta) \geq \frac{2}{45}x^4 j(x,\theta)\sin\theta \qquad (4.20)$$

where $j(x,\theta) = 15 + 2x^2\sin^6\theta - 6x^2\sin^4\theta + 4/7x^4\sin^6\theta - 1/7x^4\sin^8\theta - 2/63x^6\sin^8\theta$. Note that $j(x,\theta)$ is symmetric about $\pi/2$. Differentiating $j(x,\theta)$ with respect to θ yields $4/7x^2\sin^3\theta\cos\theta(21\sin^2\theta - 42 + 6x^2\sin^2\theta - 2x^2\sin^4\theta - 4/9x^4\sin^4\theta)$. The first factor is seen to be zero only at $\theta \in \{0, \pi/2, \pi\}$. The second factor is quadratic in x^2 and has no real roots as $(3 - \sin^2\theta)^2 + 28/3(\sin^2\theta - 2) < 0$ for all θ. Hence $j(x,\theta)$ takes stationary values for $\theta \in \{0, \pi/2, \pi\}$. Now $j(x,0) = j(x,\pi) = 15$, while

$$j(x,\pi/2) = 15 - 4x^2 + \frac{3}{7}x^4 - \frac{2}{63}x^6$$

is a cubic in x^2 which is easily shown to be decreasing with x. $j(2.4,\pi/2) > 0$ and thus for $x < 2.4$, $\frac{dh}{d\theta} \geq 0$ and h is increasing with θ.

We now need to find an increasing lower bound for h when $x \geq 2.4$. For $\theta \in [0,\pi/2]$, so that $\cos\theta \geq 0$, we can get a lower bound for h by using $\cos(2bc\sin\theta) \geq 1 - 2b^2c^2\sin^2\theta$ and $\sin(2bc\sin\theta) \geq -2bc\sin\theta$. Let this lower bound be k_1. Note that h and k_1 coincide

at $\theta = 0$. We have that

$$\frac{dk_1}{d\theta} = 16bc\sin\theta e^{-(b^2+c^2+w^2)} S_1(bc, bc\cos\theta) \tag{4.21}$$

where $S_1(x,y) = \sinh^2 y - 2y + (2+y)(x^2-y^2) - y^2$. For $x \geq y$ we have that $S_1(x,y) \geq \sinh^2 y - 2y - y^2 > 0$ for $y > 1.65$. On the other hand $S_1(x,y) > -2y + 2(x^2-y^2) > 0$ if $y < (-1+\sqrt{1+4x^2})/2$. Combining these two results we have $S_1(x,y) \geq 0$ for all $y \leq x$ if $x > 2.1$.

For $\theta \in [\pi/2, \pi]$ we have that $\cos\theta \leq 0$. We use the same bounds in h as before except we bound $\cos(2bc\sin\theta)$ by 1 in the term containing $\cos\theta$. Let this bound be k_2. Note that k_1 and k_2 coincide at $\theta = \pi/2$. It is simple to see that

$$\frac{dk_2}{d\theta} = 8bc\sin\theta e^{-(b^2+c^2+w^2)} S_2(b,c,\theta) \tag{4.22}$$

where $S_2(b,c,\theta) = 2\sinh^2(bc\cos\theta) - 4bc\cos\theta + 3b^2c^2\sin^2\theta \geq 0$. Hence k_2 is increasing with θ.

We have shown, when $x \geq 2.4$, that for $\theta \in [0,\pi/2]$, $k_1 \leq h$ is increasing, and for $\theta \in [\pi/2,\pi]$, $k_2 \leq h$ is increasing. We have also noted that $h = k_1$ at $\theta = 0$ and that $k_1 = k_2$ at $\theta = \pi/2$. We have seen that for $x < 2.4$, h is increasing. Therefore it remains to check that h is strictly positive at $\theta = 0$. Making the change of variables $s = 2b^2$ and $r = 2b(c-b)$ we have that

$$h(\sqrt{s/2}, (r/s+1)\sqrt{s/2}, 0) = \frac{s}{r^2} e^s e^{r^2/s}\, \tilde{h}(s,r)$$

where

$$\begin{aligned}\tilde{h}(s,r) \quad = \quad & \frac{(e^{r^2/s}-1)\, s\, (e^s-1-s)}{r^2} + 2\frac{e^{-r}\, s\, (1-e^{-r})}{r} \\ & \qquad - e^{-2r}\,(1-e^{-s}) - \frac{s\,(1+s)\,(1-e^{-r})^2}{r^2}.\end{aligned}$$

Let k_3 be the lower bound obtained by replacing the first term in $\tilde{h}$ by $(1+r^2/2s)(e^s-1-s)$. We differentiate k_3 with respect to r and write it as a power series expansion in r;

$$\begin{aligned} s\,r^3\,e^{2r}\,\frac{dk_3}{dr} &= 2s(1-e^{-s}-s+\frac{s^2}{2})r^3+(e^s-1-s-\frac{s^2}{2}+\frac{5s^3}{6})r^4 \\ &\quad + 2s^2\sum_{n=5}^{\infty}\frac{r^n}{n!}\Big(s(2^n-n-2)+2^n-n^2-n-2\Big) \\ &\quad + (e^s-1-s)\sum_{n=5}^{\infty}\frac{2^{n-4}r^n}{(n-4)!}. \end{aligned}$$

By using the bound $1-e^{-s}\leq s$ it is easy to see that the first term is increasing and thus positive. $e^s-1-s-\frac{s^2}{2}+\frac{5s^3}{6}\geq 0$ and $s(2^n-n-2)+2^n-n^2-n-2\geq 0$ for $n\geq 5$. Thus k_3 is increasing with r.

Finally we need to check that the $\lim_{r\to 0}k_3(s,r)>0$.

$$\lim_{r\to 0}k_3(s,r)=2(\cosh s-1-\frac{s^2}{2})>0$$

for $s>0$ and therefore $h(b,c,0)>0$ for all non-zero b, c and the Lemma is proved.

$\square$

Recalling (4.8) we can write,

$$\tilde{M}_\Lambda^\lambda=\sum_i M_i \qquad \text{and} \qquad (\tilde{M}_\Lambda^\lambda)^{-1}=\sum_i(M_i)^{-1}.$$

Let $\mathcal{E}=\sup_i\|M_i^{-1}\|$, then $\|(\tilde{M}_\Lambda^\lambda)^{-1}\|\leq\mathcal{E}$.

Lemma 4.8: *Let $\delta_\kappa=\frac{1}{2}e^{-\kappa/64}$. If $\theta>39$ and $q<\theta/13-3$ then there exists Q_4 such that for all $L>Q_4$, $\kappa>64\ln(2L^\theta)$, for all $E\in(-\delta_\kappa,\delta_\kappa)$, all λ with $|\lambda|<e^{-L^\beta/2}$,*

$$\mathbb{P}\left(d(E,\sigma(M_{\Lambda_L}^\lambda))<e^{-L^\beta}\right)<\frac{1}{L^q}.$$

Proof: If $d_i = \dim \mathcal{P}_i$ then we have

$$\|M_i^{-1}\| \leq \frac{c_{d_i}}{|\det M_i|} \|M_i\|^{d_i - 1} \tag{4.23}$$

where c_{d_i} is a constant. Obviously $d_i \in \{1, 2, 3, 4\}$, as a maximal cluster contains 4 impurities. Now, from the previous lemma we have a lower bound for $|\det M_i|$ for the case where $\lambda = 0$. Using the bound $1 - \exp(-\kappa x^2) \geq x^2 \exp(-x^2)$ for $\kappa > 1$, we can write when $\lambda = 0$,

$$|\det M_i| \geq \quad C \prod_{m<n:\, \zeta_m,\, \zeta_n \in \mathcal{C}_i} |\zeta_m - \zeta_n|^2 e^{-|\zeta_m - \zeta_n|^2}. \tag{4.24}$$

Let $u = \theta/13$. Then if $|\zeta_n - \zeta_{n'}| > 2/L^u$ for $\zeta_n, \zeta_{n'} \in \mathcal{C}_i$ with $\zeta_n \neq \zeta_{n'}$,

$$|\det M_i| \geq C \left(\frac{4e^{-(3/8)^2}}{L^{2u}} \right)^6 = C' L^{-12u}.$$

Also, if $\lambda = 0$ then

$$\|M_i\|^{d_i - 1} \leq \left(\sum_{\{n,m:\zeta_n,\zeta_m \in \mathcal{C}_i\}} |\langle m|M_i|n\rangle| \right)^{d_i - 1} < A,$$

for some constant A, independent of ζ. So for $\lambda = 0$, $\|M_i^{-1}\| \leq C'' L^{12u}$.

Therefore if D is the diagonal matrix made up of the elements $\frac{\lambda}{\omega_n}$ with $|\frac{\lambda}{\omega_n}| < e^{-L^\beta/4}$ for $\{n : \zeta_n \in \mathcal{C}_i\}$ then for L sufficiently large, by the resolvent identity

$$\mathcal{E} = \sup_i \|M_i^{-1}\| \leq \sup_i \frac{\|M_i^{-1}|_{\lambda=0}\|}{1 - \|D\| \, \|M_i^{-1}|_{\lambda=0}\|} \leq L^\theta. \tag{4.25}$$

The probability for this to occur is greater than $\mathbb{P}(|\omega_n| > e^{-L^\beta/4}$ and $|\zeta_n - \zeta_{n'}| > \frac{2}{L^u}$ for all n, n' such that $\zeta_n, \zeta_{n'} \in \mathcal{C}_i$ with $\zeta_n \neq \zeta_{n'})$ which is greater than $(1 - 2\rho_b e^{-L^\beta/4})^4 (1 - L^{3-u}) > 1 - L^{-q}$ for L sufficiently large.

Let $\delta M_\Lambda^\lambda = M_\Lambda^\lambda - \tilde{M}_\Lambda^\lambda$. Then

$$\langle n|\delta M_\Lambda^\lambda|n'\rangle = \begin{cases} 0 & \text{if } \zeta_n, \zeta_{n'} \text{ are in the same cluster,} \\ \frac{\pi}{2\kappa} P_0(\zeta_n, \zeta_{n'}) & \text{otherwise.} \end{cases}$$

Since $\kappa \geq \pi/2$ we have,

$$\|\delta M_\Lambda^\lambda\| \leq e^{-\kappa/64}.$$

¿From the resolvent identity we get

$$\|(M_\Lambda^\lambda)^{-1}\| \leq \|(\tilde{M}_\Lambda^\lambda)^{-1}\| + \|(\tilde{M}_\Lambda^\lambda)^{-1}\| \, \|\delta M_\Lambda^\lambda\| \, \|(M_\Lambda^\lambda)^{-1}\|. \tag{4.26}$$

If we can make $\|(\tilde{M}_\Lambda^\lambda)^{-1}\| \, \|\delta M_\Lambda^\lambda\| \leq \mathcal{E} e^{-\kappa/64} \leq \frac{1}{2}$ then we get,

$$\|(M_\Lambda^\lambda)^{-1}\| \leq 2\|(\tilde{M}_\Lambda^\lambda)^{-1}\|. \tag{4.27}$$

Thus, we have that if $\kappa > 64 \ln(2L^\theta)$,

$$\|(M_\Lambda^\lambda)^{-1}\| \leq 2L^\theta,$$

with a probability greater than $1 - L^{-q}$ if L is sufficiently large.

If $|E| < \frac{1}{2} e^{-\kappa/64}$ then $|E| < 1/4L^\theta$. So $\|(M_\Lambda^\lambda - E)^{-1}\| < [\|(M_\Lambda^\lambda)^{-1}\|^{-1} - 1/4L^\theta]^{-1}$. Hence from above,

$$\mathbb{P}\left(\|(M_\Lambda^\lambda - E)^{-1}\| \leq 4L^\theta\right) > 1 - \frac{1}{L^q}.$$

Now, as $E \in \mathbb{R}$,

$$d(E, \sigma(M_\Lambda^\lambda)) = \|(M_\Lambda^\lambda - E)^{-1}\|^{-1},$$

which gives us that

$$\mathbb{P}\left(d(E, \sigma(M_\Lambda^\lambda)) \leq \frac{1}{4L^\theta}\right) < \frac{1}{L^q}.$$

Taking Q_4 sufficiently large so that in addition $L^\beta > \ln(4L^\theta)$ we obtain the result.

$\square$

Lemma 4.9: *There exists Q_5 such that for all $L > Q_5$, for $q > 0$, any $E \in \mathbb{R}$, all $\kappa < L^\beta/20$, all λ with $|\lambda| < e^{-L^\beta/2}$,*

$$\mathbb{P}(d(E, \sigma(M^\lambda_{\Lambda_L})) < e^{-L^\beta}) < \frac{1}{L^q}. \tag{4.28}$$

Proof: We divide up the points of $\Lambda \cap \mathbb{Z}[i]$ into adjacent pairs $\{n_i, n_i'\}$.

Let Q_i be the two dimensional projection onto the space spanned by $|n_i\rangle$ and $|n_i'\rangle$. Let

$$U_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{-2i\kappa\zeta_{n_i}\wedge\zeta_{n_i'}} \\ 1 & e^{-2i\kappa\zeta_{n_i}\wedge\zeta_{n_i'}} \end{pmatrix} \quad , \quad U_i^* = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -e^{2i\kappa\zeta_{n_i}\wedge\zeta_{n_i'}} & e^{2i\kappa\zeta_{n_i}\wedge\zeta_{n_i'}} \end{pmatrix}$$

Let $U = \sum_i Q_i U_i Q_i^*$. Then we have, for n_i, n_i' in a pair,

$$\begin{aligned} \langle n_i|UM^0U^*|n_i\rangle &= 1 + e^{-\kappa|\zeta_{n_i}-\zeta_{n_i'}|^2} \\ \langle n_i'|UM^0U^*|n_i'\rangle &= 1 - e^{-\kappa|\zeta_{n_i}-\zeta_{n_i'}|^2} \end{aligned}$$

Now, if $r = |\zeta_{n_i} - \zeta_{n_i'}|$, then

$$\begin{aligned} \mathbb{P}[r \in (a,b)] &= \int d^2\zeta_1 \int d^2\zeta_2 r(\zeta_1)\, r(\zeta_2)\, 1_{\{|\zeta_1-\zeta_2|\in(a,b)\}} \\ &\le \int r(\zeta_1) d^2(\zeta_1) \int_{\mathbb{R}^2} r(\zeta_2) 1_{\{|\zeta_1-\zeta_2|\in(a,b)\}} d^2(\zeta_2) \\ &\le 2\pi r_b \int_a^b r dr = \pi r_b(b^2 - a^2) = \int_a^b \tilde{\rho}(r) dr \end{aligned}$$

with $\tilde{\rho}(r) = 2\pi r_b\, r$.

Let $s = e^{-\kappa r^2}$ and $e^{-5\kappa} < a < b < 1$. Then $-(\pi r_b\, ds)/(\kappa s) = \tilde{\rho}(r) dr$

$$\mathbb{P}[s \in (a,b)] < -\frac{\pi r_b}{\kappa} \int_a^b \frac{ds}{s} < \frac{\pi r_b}{\kappa} e^{5\kappa} \int_a^b ds = \frac{\pi r_b}{\kappa} e^{5\kappa}(b-a).$$

The density of $x_{nn} = \langle n|UM^0U^*|n\rangle$ is bounded by $\pi r_b e^{5\kappa}/\kappa$. So the diagonal terms $1 \pm s$ have the same bound.

For Borel subsets B of $\mathbb{R}$ let $\sigma_n^\Lambda(B) = \langle n|E_\Lambda(B)|n\rangle$, where $E_\Lambda(B)$ are the spectral projections of UM^0U^*, with $|\lambda| < e^{-L^\beta/2}$. Then by Lemma VIII.1.8 in Ref. 14,

$$\mathbb{E}_{x_{nn}}\sigma_n^\Lambda(B) < \frac{\pi r_b e^{5\kappa}}{\kappa}\int_B dx$$

and therefore

$$\mathbb{E}\sigma_n^\Lambda(B) < \frac{\pi r_b e^{5\kappa}}{\kappa}\int_B dx.$$

As in Proposition VIII.4.11 of Ref. 14, it then follows that for all $E \in \mathbb{R}$ and $\epsilon > 0$,

$$\mathbb{P}(d(E, \sigma(M^0_{\Lambda_L})) < \epsilon) < 2\frac{\pi r_b e^{5\kappa}}{\kappa}\epsilon|\Lambda_L| < 2\frac{\pi r_b e^{5\kappa}}{\kappa}\epsilon(L+1)^2 < 8\frac{\pi r_b e^{5\kappa}}{\kappa}L^2\epsilon. \tag{4.29}$$

if $L \geq 1$. Now, it is easily seen that $\sigma(M^\lambda) \subset \{ z : d(z, \sigma(M^0)) < \|M^\lambda - M^0\|\}$. Hence

$$d(E, \sigma(M^\lambda)) \geq d(E, \sigma(M^0)) - \|M^\lambda - M^0\|.$$

We can show that $\|M^\lambda - M^0\| < e^{-L^\beta/4}$ with a probability $\mathbb{P}(|\omega_n| > e^{-L^\beta/4} \quad \text{for all } n \in \Lambda_L\ ) > (1 - 2\rho_b e^{-L^\beta/4})^{(L+1)^2} > 1 - L^{-2q}$ for all $L > Q_5$ with Q_5 sufficiently large. Hence if $d(E, \sigma(M^0_{\Lambda_L})) > \epsilon + e^{-L^\beta/4}$ then $d(E, \sigma(M^\lambda_{\Lambda_L})) > \epsilon$.

So for λ with $|\lambda| < e^{-L^\beta/2}$, all $E \in \mathbb{R}$ and for all $L > Q_5$,

$$\mathbb{P}(d(E, \sigma(M^\lambda_{\Lambda_L})) > \epsilon) > \left(1 - 8\frac{\pi}{\kappa}r_b L^2(\epsilon + e^{-L^\beta/4})e^{5\kappa}\right)\left(1 - \frac{1}{L^{2q}}\right). \tag{4.30}$$

where we have used $\mathbb{P}(A) \geq \mathbb{P}(A|B)\mathbb{P}(B)$. If we have that $L^\beta > 20\kappa$ the lemma is proved.

$\square$

Finally, we can bring the two regimes for λ together to prove that:

Lemma 4.10: *There exists Q_6 and δ_κ such that for any $q > 0$, any $\lambda \in \mathbb{R}$, $E \in (-\delta_\kappa, \delta_\kappa)$, for all $L > Q_6$,*

$$\mathbb{P}(d(E, \sigma(M^\lambda_{\Lambda_{L(0)}})) < e^{-L^\beta}) < \frac{1}{L^q}.$$

Proof: Choose $\theta > 13(q+3)$. Take Q_6 larger than Q_5 such that

$$8\rho_b R^2 L^2 e^{-L^\beta/2} < \frac{1}{L^q}, \quad \text{and} \quad L^\beta > 2^8.5\ln(2L^\theta)$$

for all $L > Q_6$ and take $\delta_\kappa = \frac{1}{2}e^{-\kappa/64}$. Let $E \in (-\delta_\kappa, \delta_\kappa)$ and $|\lambda| < e^{-L^\beta/2}$. Then by Lemmas 4.8 and 4.9, for all $L > Q_6$,

$$\mathbb{P}(d(E, \sigma(M^\lambda_{\Lambda_L(0)})) < e^{-L^\beta}) < \frac{1}{L^q}.$$

On the other hand if $|\lambda| \geq e^{-L^\beta/2}$ then by Lemma 4.5, for all $L > Q_6$ we also have the above inequality.

$\square$

We finally check that the conditions of Theorem 3.4 are satisfied for $p = 3$, $q = 25$. By Lemma 4.10 condition (P2) is satisfied for $L > Q_6$ with $\eta = \delta_\kappa$ where $\delta_\kappa = \frac{1}{2}e^{-\kappa/64}$. Also from Lemma 4.10 (RA) of (P1) is satisfied with probability greater than $1 - \frac{1}{L^q}$ for $L > Q_6$.

Now in Lemma 4.4 put $u = 7$, $\gamma = \frac{1}{2}$ and let L_0 be greater than $Q_3 \vee Q_6$ and such that for any fixed $s \in (\frac{1}{2}, 1)$ (as in the regularity condition) we have

$$\frac{1}{L_0^{25}} + \frac{2}{L_0^4} < \frac{1}{L_0^3}, \qquad L_0^{14}(L_0 - L_0^s)^{1/2} > 64\ln(2L_0^7), \quad \text{and} \quad L_0^7(L_0 - L_0^s)^{1/2} > 32L_0.$$

If we choose $\kappa_0 = L_0^{4u}/4$ then we have that

$$|\langle n|\Gamma^\lambda_{\Lambda_{L_0}}(0)|n'\rangle| \leq e^{-\kappa^{1/4}L_0}$$

for all $\kappa > \kappa_0$, all n, $n' \in \Lambda_{L_0}$ with a probability greater than $1 - 2/L_0^4$. Therefore if we

take $m_0 = \kappa^{1/4}$,

$$\mathbb{P}\{\Lambda_{L_0}(0) \text{ is } (m_0, 0) - \text{regular } \} \geq 1 - \frac{1}{L_0^3}$$

where we have used that $\mathbb{P}(A \cap B) \geq 1 - \mathbb{P}(A^c) - \mathbb{P}(B^c)$ and condition (P1) is checked.

V. Proof of Theorem 3.2 parts (b) and (c).

In this section we denote by $\mathcal{L}$ the Lebesgue measure.

¿From Theorem 3.4, equation 3.5 and an application of Fubini's Theorem we can deduce that with probability one and for $\mathcal{L}$-a.e. λ, if λ is a non-zero generalized eigenvalue of $H(\omega, \zeta)$ then the corresponding eigenfunction decays exponentially. An immediate consequence is that $\sigma_{\text{ac}}\ (H) = \emptyset$. However this does not rule out the existence of singular continuous spectrum. To exclude this we need to show exponential decay for a.e. λ with respect to the spectral measure of $H(\omega, \zeta)$. We will use the ideas of Delyon, Lévy and Souillard[17].

Let $\Lambda \subset \mathbb{Z}[i]$ with $|\Lambda| = N$ as before and define the restriction of H to Λ by,

$$H_\Lambda = \sum_{n \in \Lambda} \omega_n |f_{\zeta_n}\rangle\langle f_{\zeta_n}|.$$

If ψ_k are eigenfunctions of H_Λ with eigenvalues λ_k, $k = 1, \ldots, N$ and $\|\psi_k\| = 1$ for all k, then

$$H_\Lambda \psi_k = \lambda_k \psi_k.$$

Define the resolution of the identity of the restriction of H to H_Λ by

$$E_\Lambda(B) = \sum_{k: \lambda_k \in B} |\psi_k\rangle\langle\psi_k|\ ,$$

where B is a Borel subset of $\mathbb{R}$ and let $\sigma_\Lambda^{\phi,\phi} = \langle\phi|E_\Lambda|\phi\rangle$ for some $\phi \in \mathcal{H}_0$. For $\Lambda \nearrow \mathbb{Z}[i]$, $\sigma_\Lambda^{\phi,\phi}$ converges weakly as a measure to $\sigma^{\phi,\phi}$, the spectral measure of H. We can write $\sigma_\Lambda^{\phi,\phi}$ explicitly:

$$\sigma_\Lambda^{\phi,\phi}(B) = \sum_{k:\lambda_k \in B} |\langle\phi|\psi_k\rangle|^2 \,. \tag{5.1}$$

As in Section III the eigenvalues λ_k of H_Λ with eigenfunction ψ_k must satisfy $M_\Lambda^\lambda \xi_k = 0$ where $\xi_k(n) = \sqrt{2\kappa/\pi}\ \omega_n \langle f_{\zeta_n}|\psi_k\rangle$. Thus we can expect N solutions of λ for $\det M_\Lambda^\lambda = 0$. We will look at solutions as a function of one of the ω_n only.

Using $\frac{\partial H_\Lambda}{\partial \omega_n} = |f_{\zeta_n}\rangle\langle f_{\zeta_n}|$, a calculation of $\langle\psi_k|d/d\omega_n(H_\Lambda\psi_k)\rangle$ yields that

$$\frac{d\lambda_k}{d\omega_n} = |\langle f_{\zeta_n}|\psi_k\rangle|^2 \,. \tag{5.2}$$

Note that if $\langle f_{\zeta_n}|\psi_k\rangle = 0$ then ψ_k remains an eigenvector for λ_k as ω_n varies and does not contribute to $\sigma_\Lambda^{f_{\zeta_n},f_{\zeta_n}}$. Also, if λ_k is degenerate, we can choose the corresponding orthonormal set of eigenvectors so that only one satisfies $\langle f_{\zeta_n}|\psi_k\rangle \neq 0$. From (5.2) we see that each λ_k is a monotonic increasing function of ω_n and from (3.4) that we get $N-1$ solutions of λ_k which are identical as $\omega_n \to \pm\infty$. The N-th solution corresponds to the ψ_k which tends to f_{ζ_n} and this value increases from $\lambda = -\infty$ at $\omega_n = -\infty$ to the lowest λ_k at $\omega_n = +\infty$ (respectively increases from the highest λ_k at $\omega_n = -\infty$ to $\lambda = +\infty$ at $\omega_n = +\infty$) as can be seen from the following argument. We would like to know the behaviour when ω_n and λ both become large. If we expand the determinant we get,

$$\left(1 - \frac{\lambda}{\omega_n}\right)\left(\frac{(-1)^{N-1}}{\Pi}\lambda^{N-1} + P(\lambda)\right) + \sum_{m\neq n}^{N} \omega_m \frac{e^{-2\kappa|\zeta_m-\zeta_n|^2}(-1)^{N-2}}{\Pi}\lambda^{N-2} + Q(\lambda) = 0$$

where $\Pi = \prod_{m\neq n}^N \omega_m$, $P(\lambda)$ is a polynomial in λ of degree $N-2$ and $Q(\lambda)$ is a polynomial in λ of degree $N-3$. We thus get an expression for ω_n in terms of λ.

$$\omega_n = \frac{\lambda^N}{\lambda^{N-1} - \sum_{m\neq n}^N \omega_m e^{-2\kappa|\zeta_m-\zeta_n|^2}\lambda^{N-2} + R(\lambda)}$$

where $R(\lambda)$ is a polynomial in λ^{N-3}. Thus for λ large, $\omega_n \sim \lambda$. This is what we expect if $\psi_k \to f_{\zeta_n}$ as then we have that $d\lambda_k/d\omega_n \to 1$.

Now recalling the fact that λ is a monotonic increasing function of ω_n we see that only one among the eigenvalues λ_k crosses any λ, i.e. the range of $\lambda(\omega_n)$ is divided into disjoint open subsets O_k such that $\bigcup_k O_k = \mathbb{R}$ and each λ_k corresponds to only one of the O_k. Therefore there is only one term corresponding to such λ_k in the sum (5.1) for $\sigma_\Lambda^{f_{\zeta_n}, f_{\zeta_n}}$.

The above results along with (5.1) and (5.2) allow us to make the following change of variables:

$$\begin{aligned} \int_{\mathbb{R}\times B} \rho(\omega_n) d\omega_n \sigma_\Lambda^{f_{\zeta_n}, f_{\zeta_n}}(d\lambda) &= \int_{\mathbb{R}} \sum_{k:\lambda_k \in B} \rho(\omega_n) d\omega_n |\langle f_{\zeta_n}|\psi_k\rangle|^2 \\ &\leq \rho_b \, \#\{\lambda_k \in B\} = \rho_b \, |B|. \end{aligned} \tag{5.3}$$

Using the weak convergence of $\sigma_\Lambda^{f_{\zeta_n}, f_{\zeta_n}}$ to $\sigma^{f_{\zeta_n}, f_{\zeta_n}}$ we can therefore write

$$\int_{\mathbb{R}} \rho(\omega_n) d\omega_n \sigma^{f_{\zeta_n}, f_{\zeta_n}}(B) \leq \rho_b \, |B|, \tag{5.4}$$

and hence the ω_n-averaged spectral measure $\mathbb{E}_{\omega_n}\left(\sigma^{f_{\zeta_n}, f_{\zeta_n}}(d\lambda)\right)$ is absolutely continuous with respect to Lebesgue measure.

We now use Kotani's "trick" (see Ref. 12). In the following, $\mathcal{B}$ will represent the Borel σ-field. We will need the following lemma who's proof is elementary.

Lemma 5.1 : *Let $\{f_n\}$ be a total countable subset of normalised vectors of a Hilbert space $\mathcal{H}$ and H a self-adjoint operator on $\mathcal{H}$ with spectral projections $E(\,\cdot\,)$. Let $c_n > 0$, $\sum_n c_n < \infty$ and $\nu = \sum_n c_n \sigma^{f_n, f_n}$, where $\sigma^{f_n, f_n}(\cdot) = \langle f_n|E(\cdot)|f_n\rangle$. Then for any $B \in \mathcal{B}$, $\nu(B) = 0$ implies that $E(B) = 0$.*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space corresponding to the ω and ζ and let $\mathcal{F}_n^*$ be the sub σ-field of $\mathcal{F}$ generated by all of these variables except ω_n for some $n \in \mathbb{Z}[i]$. If $F(\omega, \zeta, \lambda)$ is a nonnegative $\mathcal{F}_n^* \otimes \mathcal{B}$ measurable function, then from Proposition VIII.1.4 in Ref. 14 we have that,

$$\Big(\mathbb{E}\{\int F(\,\cdot\,,\,\cdot\,,\lambda)\,d\lambda\} = 0\Big) \Rightarrow \Big(\int F(\omega,\zeta,\lambda)\sigma^{f_{\zeta_n},f_{\zeta_n}}(d\lambda) = 0 \ \mathbb{P}\text{-a.e. }\Big). \tag{5.5}$$

Lemma 5.2 : *For $B \in \mathcal{B}$ let $B \mapsto E(B)$ be the spectral measure of H and let $A \in \cap_{n\in\mathbb{Z}[i]}(\mathcal{F}_n^* \otimes \mathcal{B})$, then, if for a.e. $\lambda \in \mathbb{R}$ with respect to Lebesgue measure $\mathbb{E}\{1_A(\,\cdot,\,\cdot,\lambda)\} = 0$, then $\mathbb{E}\{E(\{\lambda :\ (\,\cdot,\,\cdot,\lambda) \in A\})\} = 0$.*

Proof : Let $A \in \mathcal{F}_n^* \otimes \mathcal{B}$. If for a.e. λ with respect to Lebesgue measure $\mathbb{E}\{1_A\{(\,\cdot,\,\cdot,\lambda)\} = 0$, then by Fubini's Theorem $\mathbb{E}\{\int d\lambda 1_A(\,\cdot,\,\cdot,\lambda)\} = 0$. Combining this with (5.4) we have that $\mathbb{E}\{\int 1_A(\,\cdot,\,\cdot,\lambda)\mathbb{E}_{\omega_n}(\sigma^{f_{\zeta_n},f_{\zeta_n}}(d\lambda))\} = 0$. We now use the fact that $A \in \mathcal{F}_n^* \otimes \mathcal{B}$ with (5.5) to move the expectation over ω_n outside the integral and we obtain

$$\mathbb{E}\{1_A\{(\,\cdot,\,\cdot,\lambda)\} = 0 \quad \Rightarrow \quad \mathbb{E}\{\int 1_A(\,\cdot,\,\cdot,\lambda)\sigma^{f_{\zeta_n},f_{\zeta_n}}(d\lambda)\} = 0. \tag{5.6}$$

Finally, by taking $A \in \cap_{n\in\mathbb{Z}[i]}(\mathcal{F}_n^* \otimes \mathcal{B})$ and $\nu = \sum_n c_n \sigma^{f_{\zeta_n},f_{\zeta_n}}$, where each $c_n > 0$, $\sum_n c_n < \infty$ we have that $\mathbb{E}\{\int_{\mathbb{R}} 1_A(\,\cdot,\,\cdot,\lambda)\nu(d\lambda)\} = 0$ and from Lemma 5.1 we have the result. Now we have seen at the beginning of this section that if W is the set in $\Omega \times \mathbb{R}$ defined by:

$$\begin{aligned} W = \quad & \{(\omega,\zeta,\lambda) :\ \text{the generalized eigenfunctions of } H(\omega,\zeta) \\ & \qquad\qquad \text{with eigenvalue } \lambda \text{ decay exponentially }\}, \end{aligned}$$

then Fubini's Theorem implies that W is of $\mathbb{P} \otimes \mathcal{L}$ full measure. By taking W^c as A in Lemma 5.2 we have shown that with probability one and λ-a.e. with respect to the spectral

measure, if λ is a generalized eigenvalue of H then the corresponding eigenfunctions decay exponentially and hence Theorem 3.2 parts (b) and (c) are proven.

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